Exploiting Invariant Manifolds of Memristor Circuits to Reproduce FitzHugh-Nagumo Dynamics

Giacomo Innocenti, Alberto Tesi
Dipartimento di Ingegneria dell'Informazione,
Università degli Studi di Firenze,
Via S. Marta 3, 50139, Firenze, Italy
Email: giacomo.innocenti@unifi.it, alberto.tesi@unifi.it

Mauro Di Marco, Mauro Forti
Dipartimento di Ingegneria
dell'Informazione e Scienze Matematiche,
Università di Siena,
Via Roma 56, 53100, Siena, Italy
Email: dimarco@diism.unisi.it, forti@diism.unisi.it

***Abstract*—A key feature of circuits with ideal memristors is that the state space is decomposed into infinitely many invariant manifolds where quite a rich dynamics can be displayed. In this paper the possibility of embedding known attractors into the circuit invariant manifolds is investigated. Specifically, we propose a simple RLC circuit, containing an ideal flux-controlled memristor, that is capable to replicate the dynamics of the FitzHugh-Nagumo model. It is shown that there is a one-to-one correspondence between the dynamic behaviors generated by the model for constant values of the injected current and those displayed onto the circuit invariant manifolds.**

I. Introduction

In the last decade it has been widely recognized that memristors can be employed as synapses in neuromorphic circuits. Indeed, circuits containing memristors are foreseen as candidates for new computing architectures based on in-memory and analogue computation principles [1]–[3].

A key feature of memristor circuits is the richness of their dynamics with respect to classical nonlinear RLC circuits. Such a property, which is referred to as (extreme) multistability, is connected to the fact that the state space of circuits containing ideal memristors can be decomposed into a continuum of invariant manifolds which can be parameterized by some parameters, referred to as manifold indexes, whose values depend on the initial conditions of the circuit [4]. Moreover, it has been shown that by suitably designing pulse shaped voltage/current sources it is possible to switch the circuit dynamics between different invariant manifolds and the attractors therein contained [5], [6].

The FitzHugh-Nagumo model has been introduced in the sixties to approximate the electrical characteristics of excitable cells (e.g., neurons) [7], [8]. Its dynamics can switch between oscillating and stationary solutions, depending upon the value of a constant input, namely the injected current (see, e.g., [9], [10]). Some non-autonomous memristive FitzHugh–Nagumo circuits have been proposed in the literature, also to demonstrate the (extreme) multistability feature of memristor circuits [11] as well as their possible use for synchronization purposes [12], [13].

This paper considers the FitzHugh-Nagumo model from a different point of view, namely to address whether memristor circuits can be used to reproduce its dynamics. Specifically, a simple RLC circuit containing an ideal flux-controlled memristor is introduced in Section II. Exploiting the fact that the circuit state space can be decomposed into a continuum of invariant manifolds, Section III investigates under which conditions the circuit dynamics onto any invariant manifold is exactly that generated by the FitzHugh-Nagumo model for a suitable constant value of the injected current. Some numerical simulations of the designed memristor circuit are reported in Section IV for illustration purposes. Finally, some concluding remarks are given in Section V.

II. Preliminaries and problem formulation

In this section we first recall the FitzHugh-Nagumo model and present a memristor circuit composed of a resistor, a capacitor, an inductor and a flux-controlled (ideal) memristor. Then, we formulate the problem addressed in the paper.

A. The FitzHugh-Nagumo model

The FitzHugh-Nagumo model is a two-dimensional approximation of the Hodgkin-Huxley neuron model [9], [10]. Its dynamics obeys the following nonlinear differential equations

$$\left\{ \begin{array}{rcl} \dot{V}(t) & = & f(V(t)) - W(t) + I(t) \\ \dot{W}(t) & = & a(bV(t) - cW(t)) \end{array} \right. \tag{1}$$

where V is the membrane potential, W is an internal variable, $f : \mathbb{R} \to \mathbb{R}$ is a third-order polynomial such that $f(0) = 0$, a, b, c are constant parameters and I is the injected current which constitutes the neuron stimulus and it is assumed to be constant over time, i.e., $I(t) = I_0$, $\forall t \geq t_0$. Without any loss of generality, V, I, and t can be regarded as normalized variables, i.e., not described in the biological units. If $I_0 = 0$ there is no stimulation, while for $I_0 \neq 0$ the model can display stationary and oscillating behaviors. A detailed analysis of the dynamics generated by the FitzHugh-Nagumo model by varying the constant injected current I_0 is reported in [9], [10]. In the sequel we denote by $V^0(t)$ and $W^0(t)$, $t \geq t_0$, the solutions of (1) with initial conditions $V^0(t_0) = V_0$, $W^0(t_0) = W_0$ and constant stimulus $I(t) = I_0$. Also, we assume that the polynomial $f(V)$ has the following form

$$f(V) = V - \frac{V^3}{3} \ . \tag{2}$$

However, the analysis developed in Section III applies to any third-order polynomial as well.

B. The FitzHugh-Nagumo memristor circuit

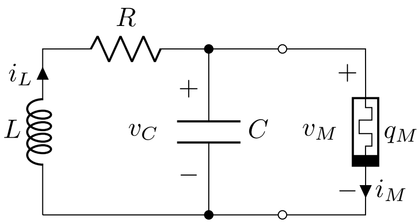

Figure 1. FitzHugh-Nagumo memristor circuit.

The memristor circuit contains a resistor R, an inductor L, a capacitor C and an ideal flux-controlled memristor, as depicted in Fig. 1. The dynamics of the circuit obeys

$$\begin{cases} \dot{v}_C(t) &= \frac{1}{C} i_L(t) - \frac{1}{C} i_M(t) \\ \dot{i}_L(t) &= -\frac{1}{L} v_C(t) - \frac{R}{L} i_L(t) \\ \dot{\varphi}_M(t) &= v_M(t) \ , \end{cases} \tag{3}$$

where v_C is the capacitor voltage, i_L is the inductor current, and v_M, i_M, φ_M and q_M are the voltage, current, flux and charge of the memristor, respectively. The flux and charge are related by a nonlinear characteristic $N : \mathbb{R} \to \mathbb{R}$, i.e.,

$$q_M = N(\varphi_M) \ , \tag{4}$$

and hence we have the relations

$$i_M(t) = \dot{q}_M(t) = \dot{N}(\varphi_M(t)) = N'(\varphi_M(t)) v_M(t) \ , \tag{5}$$

where $N'(\cdot)$ is the first order derivative of $N(\cdot)$. Similarly to (1), the variables in (3) can be intended as normalized values without any loss of generality. Ideal memristors can be implemented by electronic emulators as illustrated in [14] and in the references therein.

The next characterization of the memristor circuit invariant manifolds can be proven (see, e.g., Proposition 2 in [6]).

Proposition 1. *The invariant manifolds of the FitzHugh-Nagumo memristor circuit are*

$$\mathcal{M}_{\mathcal{J}} = \big\{ v_C, i_L, \varphi_M : \\ \varphi_M + RCv_C + Li_L + RN(\varphi_M) = \mathcal{J} \big\} \ , \tag{6}$$

where $\mathcal{J} \in \mathbb{R}$ is the manifold index. Moreover,

$$\bigcup_{\mathcal{J} \in \mathbb{R}} \mathcal{M}_{\mathcal{J}} \equiv \mathbb{R}^3 \ , \tag{7}$$

i.e., the state space of the FitzHugh-Nagumo memristor circuit is decomposed into a continuum of invariant manifolds.

Remark 1. *The above proposition implies that the solutions $\varphi_M^0(t)$, $v_C^0(t)$, $i_L^0(t)$, $t \geq t_0$, of (3)-(5), with initial conditions $\varphi_M^0(t_0) = \varphi_{M_0}$, $v_C^0(t_0) = v_{C_0}$, $i_L^0(t_0) = i_{L_0}$, are such that the scalar variable*

$$z \doteq \varphi_M + RCv_C + Li_L + RN(\varphi_M) \tag{8}$$

satisfies

$$z(t) = \mathcal{J} = \varphi_{M_0} + RCv_{C_0} + Li_{L_0} + RN(\varphi_{M_0}) \tag{9}$$

and hence $\dot{z}(t) = 0$ for all $t \geq t_0$. Note that the initial conditions φ_{M_0}, v_{C_0}, i_{L_0} dictate the invariant manifold $\mathcal{M}_{\mathcal{J}}$ where the solutions are confined to lie.

C. Problem formulation

In the next section we address the problem of how the solutions $\varphi_M^0(t)$, $v_C^0(t)$, and $i_L^0(t)$ of the memristor circuit lying on the manifold $\mathcal{M}_{\mathcal{I}}$ can be used to replicate the solutions $V^0(t)$ and $W^0(t)$ of the FitzHugh-Nagumo model.

III. Equivalence of the FitzHugh-Nagumo model and memristor circuit dynamics

We first show that $\varphi_M^0(t)$, $v_C^0(t)$, and $i_L^0(t)$ lying on the invariant manifold $\mathcal{M}_{\mathcal{I}}$ can be obtained via the solutions $x_1(t)$ and $x_2(t)$ of the following two-dimensional system

$$\begin{cases} \dot{x}_1(t) &= \frac{1}{C} x_2(t) - \frac{1}{C} N(x_1(t)) \\ \dot{x}_2(t) &= -\frac{1}{L} x_1(t) - \frac{R}{L} x_2(t) + \frac{1}{L} \mathcal{J} \ . \end{cases} \tag{10}$$

The next result holds true.

Proposition 2. *Let $\varphi_M^0(t)$, $v_C^0(t)$, and $i_L^0(t)$ be the solutions of (3)-(5) with initial conditions $\varphi_M^0(t_0) = \varphi_{M_0}$, $v_C^0(t_0) = v_{C_0}$, $i_L^0(t_0) = i_{L_0}$ lying on $\mathcal{M}_{\mathcal{I}}$. Then, for all $t \geq t_0$ we have*

$$\begin{cases} \varphi_M^0(t) &= x_1^0(t) \\ v_C^0(t) &= \frac{1}{C} \left(x_2^0(t) - N(x_1^0(t)) \right) \\ i_L^0(t) &= \frac{1}{L} \left(\mathcal{J} - x_1^0(t) - Rx_2^0(t) \right) \ , \end{cases} \tag{11}$$

where $x_1^0(t)$ and $x_2^0(t)$ are the solutions of (10) with initial conditions $x_1^0(t_0) = \varphi_{M_0}$ and $x_2^0(t_0) = Cv_{C_0} + N(\varphi_{M_0})$.

Proof. We have to verify that $\varphi_M^0(t)$, $v_C^0(t)$, and $i_L^0(t)$ in (11) satisfy (3)-(5) and belong to $\mathcal{M}_{\mathcal{I}}$ for all $t \geq t_0$. Taking into account that $v_M = v_C$, the third equation in (3) can be rewritten as $\dot{\varphi}_M(t) = v_C(t)$ and hence, exploiting the first two equations in (11), equivalently as

$$\dot{x}_1^0(t) = \frac{1}{C} \left(x_2^0(t) - N(x_1^0(t)) \right) \ ,$$

which is indeed the first equation of (10). Analogously, taking into account (5), the first equation in (3) can be rewritten as $C\dot{v}_C(t) = i_L(t) - \dot{N}(\varphi_M(t))$ and hence, exploiting the first two equations in (11), it boils down to

$$\dot{x}_2^0(t) - \dot{N}(x_1^0(t)) = \frac{1}{L} \left(\mathcal{J} - x_1^0(t) - Rx_2^0(t) \right) - \dot{N}(x_1^0(t)) \ ,$$

i.e., the second equation of (10). To show that the remaining equation $L\dot{i}_L(t) = v_C(t) - Ri_L(t)$ is satisfied for $i_L = i_L^0$ and v_C^0, it is enough to compute $\dot{i}_L^0(t)$ with $i_L^0(t)$ as in the third equation of (11). Finally, we have that z in (8) amounts to

$$z(t) = \varphi_M^0(t) + RCv_C^0(t) + Li_L^0(t) + RN(\varphi_M^0(t))$$

and hence from (11) we get $z(t) = \mathcal{J}$ for $t \geq t_0$, which completes the proof. ■

To proceed, we rewrite the FitzHugh-Nagumo model (1), with $I(t) = I_0$, $\forall t \geq t_0$, in the following equivalent form

$$\begin{cases} \dot{z}_1(t) &= z_2(t) + f(z_1(t)) \\ \dot{z}_2(t) &= -abz_1(t) - acz_2(t) + acI_0 \ , \end{cases} \tag{12}$$

where

$$\begin{cases} z_1 &\doteq V \\ z_2 &\doteq I_0 - W \ . \end{cases} \tag{13}$$

Now, we can prove the main result.

Proposition 3. *Let $V^0(t)$ and $W^0(t)$ be the solutions of (1) with initial conditions $V^0(t_0) = V_0$, $W^0(t_0) = W_0$, and $I(t) = I_0$. Then, for all $t \geq t_0$ we have*

$$\begin{cases} \varphi_M^0(t) &= V^0(t) \\ v_C^0(t) &= I_0 - W^0(t) + f(V^0(t)) \\ i_L^0(t) &= -abV^0(t) + acW^0(t) \ , \end{cases} \tag{14}$$

where $\varphi_M^0(t)$, $v_C^0(t)$, and $i_L^0(t)$ are the solutions of (3)-(5) once the circuit parameters R, L, C and the nonlinear characteristic $N(\cdot)$ are designed as

$$R = \frac{c}{b} \quad , \quad L = \frac{1}{ab} \quad , \quad C = 1 \quad , \quad N(\cdot) = -f(\cdot) \tag{15}$$

and the initial conditions φ_{M_0}, v_{C_0}, and i_{L_0} are set as

$$\varphi_{M_0} = V_0, \ v_{C_0} = I_0 - W_0 + f(V_0), \ i_{L_0} = -abV_0 + acW_0. \tag{16}$$

Proof. Clearly, if conditions (15) are satisfied and the constant current I_0 and the manifold index $\mathcal{I}$ are such that

$$b\mathcal{J} = cI_0 \ , \tag{17}$$

then the two-dimensional systems (10) and (12) are identical. This implies that

$$\begin{cases} x_1^0(t) &= V^0(t) \\ x_2^0(t) &= I_0 - W^0(t) \end{cases}$$

and thus relations (14) directly follow from (11). ∎

Remark 2. *Proposition 3 states that the solutions $V^0(t)$ and $W^0(t)$ of the FitzHugh-Nagumo model (1) with $V^0(t_0) = V_0$, $W^0(t_0) = W_0$, and $I(t) = I_0$ are obtained from $\varphi_M^0(t)$, $v_C^0(t)$, $i_L^0(t)$ satisfying (3)-(5) with R, L, C, and $N(\cdot)$ in (15) and initial conditions (16). Indeed, from the first and the third equations in (14) we get*

$$\begin{cases} V^0(t) &= \varphi_M^0(t) \\ W^0(t) &= \dfrac{b}{c}\varphi_M^0(t) + \dfrac{1}{ac} i_L^0(t) \ , \end{cases} \tag{18}$$

while the second equation and (9) yield

$$I_0 = \frac{b}{c}\varphi_{M_0} + \frac{1}{ac} i_{L_0} + v_{C_0} - f(\varphi_{M_0}) = \frac{b}{c}\mathcal{J} \ . \tag{19}$$

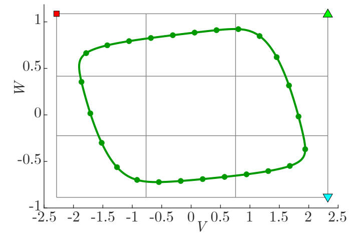

Figure 2. FitzHugh-Nagumo dynamics for $I_0 = 0.1250$ and parametric configuration $a = 0.2$, $b = 0.4$, $c = 0.32$. The model is initialized on the attractive limit cycle. The circles and the bounding box surrounding the cycle are used to highlight the relation (14) with the equivalent memristor circuit.

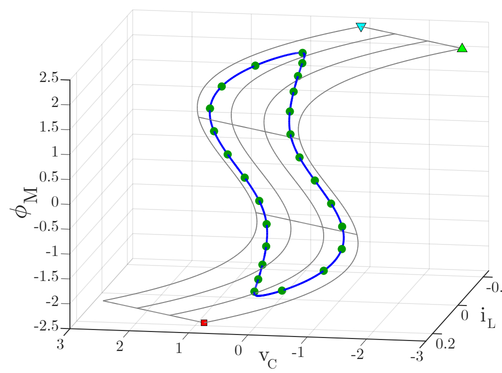

Figure 3. Structure of the memristor circuit invariant manifold that corresponds to the situation depicted in Fig. 2. The limit cycle (blue line) is numerically simulated for the initial conditions equivalent to those of the FitzHugh-Nagumo limit cycle, and it is compared with the circles which are mapped (via (14)) from those in Fig. 2. The patch surrounding the trajectory is the image of the bounding box of Fig. 2, and it provides a general idea of the invariant manifold shape.

IV. Numerical simulations

In this section we consider the FitzHugh-Nagumo model (1) with the parametric configuration

$$a = 0.2, \ b = 0.4, \ c = 0.32 \ , \tag{20}$$

which is equivalent to the original model [7]–[9]. According to (15), the memristor circuit parameters have the values

$$R = 0.8, \ L = 12.5, \ C = 1 \ . \tag{21}$$

Parameters (20) and (21) are in normalized units, and they link together the (normalized) dynamics (1) and (3). Any implementation of the circuit requires that (3) is converted into actual electric units. The model is simulated in the Matlab environment for $I_0 = 0.1250$ starting from $V_0 = -1.8932$, $W_0 = 0.4494$. After a transient, the dynamics converges towards the limit cycle depicted in Fig. 2. A bounding box surrounding the cycle is also drawn. One of the points marked on the cycle is taken to compute via (14) the equivalent initial conditions of the memristor circuit. The corresponding trajectory is depicted in Fig. 3, where both marked points on the FitzHugh-Nagumo cycle and the bounding box have been drawn after the application of (14).

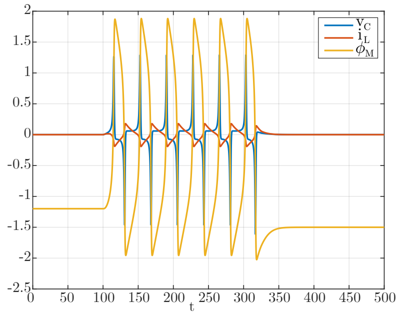

Figure 4. Time behavior of the memristor circuit state variables. From the initial equilibrium point, the circuit is brought into a periodic regime, and then back to a different fixed point.

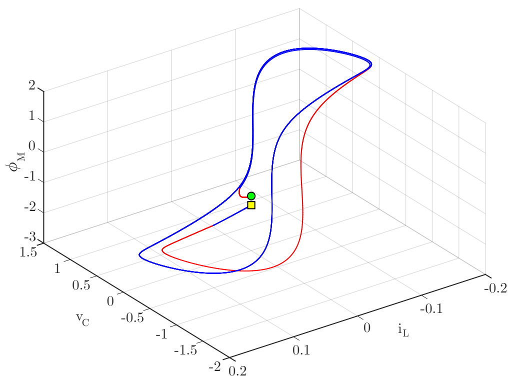

Figure 5. The circuit is initially in a fixed point condition (green circle). Then, during a transient (red line) generated by the pulse programmed voltage source u, it moves to a different invariant manifold, where the trajectory (blue line) converges to a limit cycle. A second pulse of u makes the state move back (red line) to a manifold, different from the first one, where the trajectory (blue line) is captured by a globally attractive fixed point (yellow square).

To illustrate that the memristor circuit is capable of replicating the FitzHugh-Nagumo dynamics induced by different values of I_0, an independent voltage source $u(t)$ is introduced in series to the inductor. This source can be suitably pulse programmed to modify the manifold index $\mathcal{J}$ [6], and hence I_0, according to (17). Then, the circuit is simulated with initial conditions $\varphi_{M_0} = -1.1994$, $v_{C_0} = 0$, $i_{L_0} = 0$ which correspond to $V_0 = -1.1994$, $W_0 = -1.4993$, and $I_0 = -0.8750$, according to (18) and (19). For this I_0 the dynamics has a unique globally attractive equilibrium point exactly in those initial conditions. At $t = 100$ a rectangular pulse long 10 time units and with area equal to 0.5 is injected into the circuit via the voltage source, thus steering the dynamics onto $\mathcal{M}_{\mathcal{J}}$ with $\mathcal{J}$ corresponding to $I_0 = -0.2495$. There, the FitzHugh-Nagumo model has a stable limit cycle, and, indeed, the circuit dynamics becomes periodic after a short transient. Finally, at $t = 300$ another rectangular pulse long 20 with area -1 brings back the dynamics onto $\mathcal{M}_{\mathcal{J}}$ with $\mathcal{J}$ related to $I_0 = -1.5000$, where the FitzHugh-Nagumo model has a unique globally attractive equilibrium point, different from the first one. Figures 4, 5, and 6 depict the complete experiment.

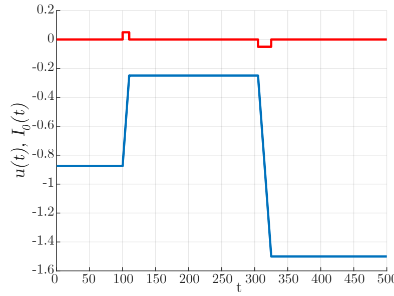

Figure 6. The upper plot (red line) depicts the pulses provided by the voltage source u. The lower plot (blue line) reports the variation of the FitzHugh-Nagumo current I_0 induced by the pulses.

V. Conclusions

In this paper it is shown that the dynamics of the FitzHugh-Nagumo model can be reproduced via a simple RLC circuit containing an ideal flux-controlled memristor. Specifically, there is a one-to-one correspondence between the dynamics of the model generated by constant values of the injected current and that displayed onto the invariant manifolds of the memristor circuit. Also, it is shown that it is possible to make the memristor circuit switching between stationary and oscillating behaviors via a pulse programmed voltage source.

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