Bioinspired Cellular Nonlinear Networks working on the edge of chaos

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***Abstract*— The most widely-used model of the excitation and propagation of impulse (action potential) in nerve membranes is the Hodgkin-Huxley system. In this paper we consider its simplification - a coupled FitzHugh Nagumo model. We shall study its dynamics from the point of view of local activity theory and we shall define the edge of chaos region in which complex behavior appears. Simulations show that oscillatory patterns, chaotic patterns, or divergent patterns may emerge if the selected cell parameters are located in locally-active domains but near the edge of chaos.**

Keywords—Hodgkin Huxley model, coupled FitzHugh Nagumo CNN, edge of chaos, pattern formation

I. INTRODUCTION

Information processing in the brain takes place in a dense network of neurons connected through synapses. The collaborative work between these two components (Synapses and Neurons) allows basic brain functions such as learning and memorization. An efficient emulation of these computational concepts is possible only by overcoming the so called von Neumann bottleneck which limits the information processing capability of conventional systems. To this end, mimicking the neuronal architectures with silicon-based circuits, on which neuromorphic engineering is based, is accompanied by the development of new devices with neuromorphic functionalities. Continued interest in bio-inspired computing will likely make resistive switching technologies an important area of research during the next decade.

For reaction-diffusion Cellular Nonlinear Networks (CNN) model, one can determine the domain of the cell parameters in order for the cells to be locally active, and thus potentially capable of exhibiting complexity. In the literature, the so called edge of chaos (EC) means a region in the parameter space of a dynamical system, where complex phenomena and information processing can emerge. The problem with such a definition is somewhat circular since it is defined in terms of phase transition regime which itself lacks a precise mathematical definition. In this paper we shall prove strong mathematical definition of the edge of chaos for bioinspired CNN model.

The Hodgkin–Huxley equations of the cardiac Purkinje fiber (CPF) model of morphogenesis in [4] describe the long-lasting action and pacemaker potentials of the Purkinje fiber of the heart for the first time. In the figure below we present the circuit implementation of Hodgkin Huxley model.

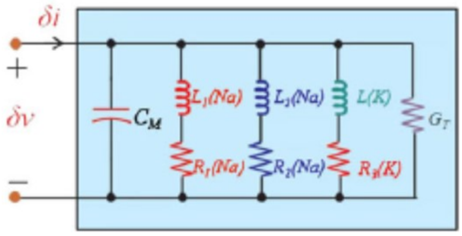


Figure 1. Small signal Hodgkin Huxley's model.

Hodgkin injected a DC-current of varying amplitude and discovered [4] that some systems could exhibit repetitive spiking with arbitrary low frequencies, while the others discharged in a narrow frequency band.

The original Hodgkin Huxley equations have the following form:

$$\frac{dV}{dt} = -\frac{1}{C_m}((400\, m^3 h + 0.14)(V - a) + 1.2\exp(-V - 90/50) + 0.015\exp((V + 90)/60) + 1.2\, n^4)(V + b)$$

$$\frac{dm}{dt} = \alpha_m(V)(1 - m) - \beta_m(V)m$$
$$\frac{dh}{dt} = \alpha_h(V)(1 - h) - \beta_h(V)h$$
$$\frac{dn}{dt} = \alpha_n(V)(1 - n) - \beta_n(V)n \qquad (1)$$

where $a = E_{N_a} = 40, b = -E_k = 100, C_m = 12$ and E_{N_a}, E_k and C_m are the sodium equilibrium potential, potassium equilibrium potential and membrane capacity, respectively. The other terms are defined as follows,

$$\alpha_m(V) = 0.1(-V - 48)/(\exp((-V - 48)/15) - 1)$$
$$\beta_m(V) = 0.12(V + 8)/(\exp((V + 8)/5) - 1)$$
$$\alpha_h(V) = 0.17(\exp((-V - 90)/20))$$
$$\beta_h(V) = 1/(\exp((-V - 42)/10) + 1)$$
$$\alpha_n(V) = 0.0001(-V - 50)/(\exp((-V - 50)/10) - 1)$$
$$\beta_n(V) = 0.002\exp((-V - 90)/80)$$

where V is equal to the membrane potential E minus the resting potential E_r ,(V is called the membrane potential). The trajectory of the original CPF equations is the same as the corresponding trajectory given in [4].

The famous Hodgkin-Huxley neuron model [4] is the first mathematical model describing neural excitation transmission derived from the angle of physics and lays the basis of electrical neurophysiology. FitzHugh Nagumo equation [3], [5], which is a simplification of Hodgkin-Huxley model, describes the generation and propagation of the nerve impulse along the giant axon of the squid. The FitzHugh Nagumo systems are of fundamental importance for understanding the qualitative nature of nerve impulse propagation. Shortly after the publication of Hodgkin and Huxley's equations (HH) for the squid giant axon [4], Richard FitzHugh [3] undertook an analysis of the mathematical properties of their equations. He used the new techniques of nonlinear mechanics. This was before digital computers became easily accessible. FitzHugh started by planning how to program an analogue computer which could be used to solve the Hodgkin-Huxley equations. With this computer he plotted solutions of the HH equations. The operation of the analogue computer required the skill of an electronic engineer as well as those of a mathematician. In this analogue computer, the variables in the HH equations are represented by voltages. Each variable was transformed into a voltage with a separate scale factor. These voltage signals were passed from one unit to another.

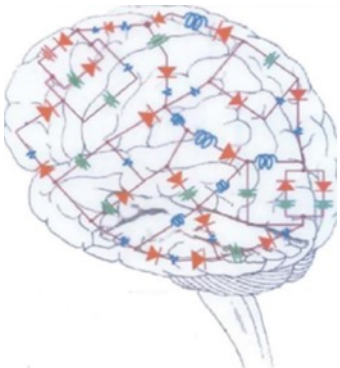


Figure 2. FitzHugh computer.

In this paper we consider the simplification of Hodgkin Huxley model - coupled FitzHugh Nagumo system. In Section 2 we introduce the model, in Section 3 we study its dynamics and determine the edge of chaos region. Section 4 deals with pattern formation in the model having application in cardiology and neurophysiology.

II. Generation and Propagation of the Nerve Impulse

Nonlinear reaction-diffusion types of equations are widely used to describe phenomena in different fields, such as the biology-Fisher model [7], the Hodgkin-Huxley model [4] and its simplification—the FitzHugh Nagumo nerve conduction model [3,5], etc.

Based on the finite propagating speed in the signal transmission between the neurons, the coupled FitzHugh Nagumo neural system was proposed [3.5]. In this paper we shall consider the following system:

$$\begin{cases} \dot{u}_1 = -u_1(u_1-1)(u_1-a) - u_2 + cf(u_3) \\ \dot{u}_2 = b(u_1 - \gamma u_2) \\ \dot{u}_3 = -u_3(u_3-1)(u_3-a) - u_4 + cf(u_1) \\ \dot{u}_4 = b(u_3 - \gamma u_4), \end{cases} \tag{2}$$

where a , b , γ are positive constants, $u_{1,2}$ represent transmission variables, and $u_{3,4}$ are receiving variables; c measures the coupling strength, $f \in C^3, f(0) = 0, f'(0) = 1$. We shall take $f(x) = \tan^{-1} x$ in our investigation. System (2) is symmetric. Thus, considering the existence, spatio-temporal patterns and stability of its Hopf bifurcation periodic solutions is interesting.

In the seminal work by Rinzel and Ermentrout [6] it was shown that the difference in behavior is due to different bifurcation mechanisms of excitability. For dynamical systems in neuroscience, the type of bifurcation determines the computational properties of neurons. Neuronal models can be excitable for some values of parameters, and fire spikes periodically for other values. These two types of dynamics correspond to a stable equilibrium and a limited cycle attractor, respectively. When the parameters change, the models can exhibit a transition from one qualitative type of dynamics to another. Thus, the stability and bifurcation of neural network systems attract a lot of attention. At the same time, information transmission among neurons is carried out through synapses, and therefore the coupling among neurons is also achieved through synapses. Coupling among neurons can be classified into a gap junction and a chemical synapse coupling. Chaos and bifurcations can occur even in most simple systems, and moreover, coupled neurons could synchronize and exhibit collective behavior.

For a coupled Fitzhugh Nagumo system (2), we shall introduce CNN architecture. In our model each cell will be arranged on a two-dimensional square grid and will be connected to adjacent cells through coupling devices that mimic 2-D spatial diffusion and transmit the cell's state to its neighboring cells, as in conventional CNN. Then the CNN model under consideration will be the following:

$$\begin{cases} \frac{du_j^1}{dt} = -u_j^1(u_j^1-1)(u_j^1-a) - u_j^2 + cf(u_j^3) \\ \frac{du_j^2}{dt} = b(u_j^1 - \gamma u_j^2) \\ \frac{du_j^3}{dt} = -u_j^3(u_j^3-1)(u_j^3-a) - u_j^4 + cf(u_j^1) \\ \frac{du_j^4}{dt} = b(u_j^3 - \gamma u_j^4), j = 1, \dots, n. \end{cases} \tag{4}$$

The system (2) is transformed into a system of ordinary differential equations which is identified as the state equation of CNN with appropriate templates. We map the variables u_1, u_2, u_3 and u_4 into CNN layers such that the state voltage of a CNN cell at a grid point is $u_j^i, i = 1,2,3,4, n = M.M$; M is number of the cells. The original sigmoid output circuit will be eliminated to further reduce the size of the processing elements as well as to improve the speed of computation. Therefore, the proposed architecture is more compact and versatile, as well as suitable for a VLSI implementation. The advantages of new proposed architecture are high density, non-volatility, and programmability of synaptic weights.

III. Edge of Chaos in Memristor CNN Model (4)

Let us consider now the CNN model (4) of the coupled FitzHugh Nagumo neural system (2). First, we shall find the equilibrium points of (4). According to the theory of dynamical systems the equilibrium points $\hat{u}_j^i$ of (4) are these for which:

$$-u_j^1(u_j^1-1)(u_j^1-a) - u_j^2 + c\tan^{-1}(u_j^3) = 0$$

$b(u_j^1 - \gamma u_j^2 = 0$

$-u_j^3(u_j^3 - 1)(u_j^3 - a) - u_j^4 + c\tan^{-1}(u_j^1) = 0$

$$b(u_j^3 - \gamma u_j^4) = 0 \quad (5)$$

System (5) may have one, two, three or four real roots $\hat{u}_j^1$, $\hat{u}_j^2$, $\hat{u}_j^3$, $\hat{u}_j^4$, respectively. In general, these roots are functions of the cell parameters (a, b, c, γ). In other words, we have $\hat{u}_j^i = \hat{u}_j^i(a, b, c, \gamma)$, $i = 1,2,3,4$. We shall consider only the equilibrium point $E_0 = (0,0,0,0)$.

We shall now calculate the Jacobian matrix of (5) about equilibrium point E_0. In our particular case the associate linear system in a sufficient small neighbourhood of the equilibrium point E_0 can be given by

$$\frac{dz}{dt} = DF(E_0)z,$$

$DF(E_0) = J$ is the Jacobian matrix of each of the equilibrium points and can be computed by:

$$J_{p,s} = \frac{\partial F_p}{\partial u_s}|_{u=E_0}, 1 \le p, s \le n. \quad (6)$$

In our particular case the Jacobian matrix in the equilibrium point E_0 is:

$$J = \begin{bmatrix} -a & -1 & c & 0 \\ b & -b\gamma & 0 & 0 \\ c & 0 & -a & -1 \\ 0 & 0 & b & -b\gamma \end{bmatrix}$$

We shall calculate the trace $Tr(E_0) = \sum_{q=1}^{N} \lambda_q$. In the equilibrium point $E_0 = (0,0,0,0)$ the trace is $Tr(0,0,0,0) = -2(a + b\gamma)$.

We shall identify the cell state variables u_j as follows: u_j is associated with the node-to-datum voltage at node (j) of a two-dimensional grid G of linear resistors.

Definition 1

Stable and Locally Active Region $SLAR(E)$ at the equilibrium point E_0 for the CNN model (4) is such that $Tr < 0$.

In our particular case we have: $Tr(0,0,0,0) = -2(a + b\gamma) < 0$ for all a, b, c, γ positive. Therefore in the equilibrium point $E_0 = (0,0,0,0)$ we have a stable and locally active region.

We shall identify the edge of chaos domain (EC) in the cell parameter space of the CNN model (4) by using the following definition [2]:

Definition 2

The CNN model (4) is said to be operating in the edge of chaos (EC) regime if and only if there is at least one equilibrium point E_0, which belongs to $SLAR(E)$ according to definition 1.

The following theorem then holds:

Theorem 1

CNN model (4) of the coupled FitzHugh Nagumo system (2) is operating in the edge of chaos regime for all a, b, c and γ positive. For these parameter values there is at least one equilibrium point which belongs to $SLAR(E)$.

Proof:

After solving (5) we find that one of the equilibrium points is $E_0 = (0,0,0,0)$. Then we check the conditions for local activity and stability given by Definition 1. The results show that the equilibrium point $E_0 = (0,0,0,0)$ satisfies these conditions for the parameter set $a, b, c, \gamma > 0$. Therefore, there is at least one equilibrium point which is both locally active and stable. According to Definition 2, this means that the CNN model (4) is operating in the edge of chaos regime [2]. The theorem is proved.

The simulations of the edge of chaos EC in which is operating the CNN model (4) are given in Fig. 3:

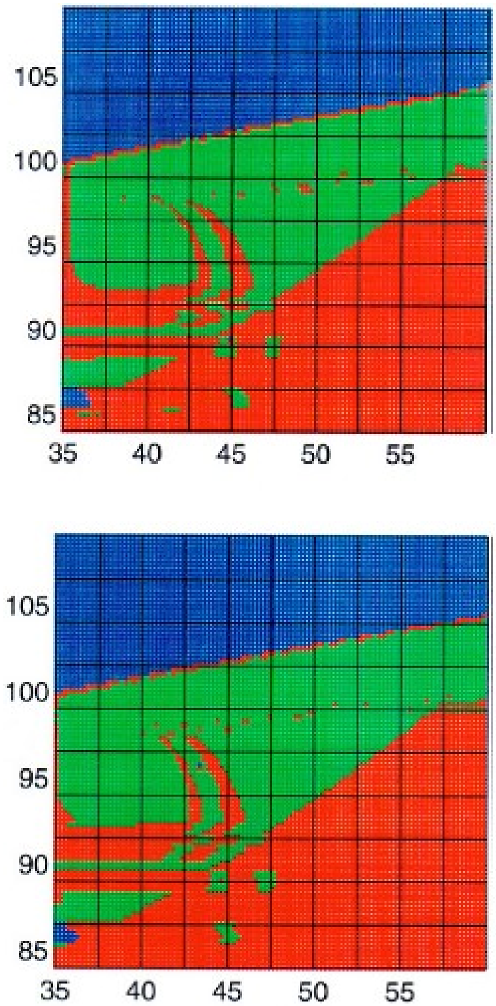


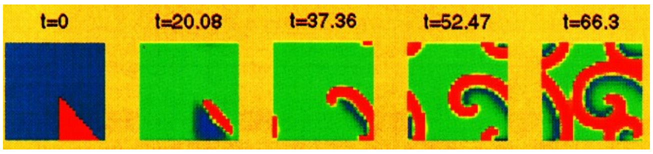
Figure 3. Bifurcation diagram of (4). Edge of chaos (red), locally-active unstable domain (green), and locally-passive domain (blue).

Remark 1. In the edge of chaos region different patterns occur such as oscillatory patterns, convergent (static) patterns, and divergent (unbounded). In particular, the emergence of complex patterns may occur if the corresponding cell parameters are chosen in the locally-active unstable domain, but near the edge of chaos domains. In summary, the above investigation confirms once again that the local activity theory provides a practical and explicit analytical tool for determining a subset of the cell parameter space where complexity may emerge.

IV. Pattern formation

For our CPF (1) computer simulations show that oscillatory patterns, chaotic patterns, or divergent patterns may emerge if the selected cell parameters are located in locally-active domains but near the edge of chaos. This research demonstrates once again the effectiveness of the local activity theory in choosing the parameters for the emergence of complex (static and dynamic) patterns in a homogeneous lattice formed by coupled locally-active cells. It is interesting to find that the cell parameter of a normal heart is located in the locally-active unstable domain but near an edge of chaos domain. Roughly speaking, our computer simulation shows as the values of E_{N_a} and E_K are increased, the frequency of the heartbeat (corresponding to the periodic frequency of the membrane potential described via the CPF equations) also increases. However, the amplitude of the membrane potential decreases until the heart stops beating. Conversely as the values of E_{N_a} and E_K are decreased, the frequency of the heartbeat is also decreased until the heart stops beating. These phenomena can be well explained via the corresponding bifurcation diagrams. Extensive computer simulations on the figure below show that if the chosen cell parameters are near the edge of chaos and are located in a locally-active unstable region, the corresponding patterns may show chaotic, periodic, or unbounded characteristics.

(a) irregular spatio-temporal pattern



(b) spiral wave dynamic pattern

(c) edges of the associated initial image

Figure 3. Dynamic computer simulations at different cell parameter points.

Remark 2. Many processes observed in nature can be described as a hierarchy of homogeneous interactions between many identical cells. Such cells may consist of molecules, physical devices, electronic circuits, biological cells, dynamical systems, artificial life-like cells, and other abstract entities. What is characteristic of all such systems is that under certain circumstances, collective complexity may emerge, i.e., the function of the entire system is more than simply summing up the functions of its parts. Life itself is a supreme manifestation of complexity [2].

V. Conclusions

In this paper we study coupled FitzHugh Nagumo system which a simplification of Hodgkin–Huxley equations of the cardiac Purkinje fiber (CPF) model of morphogenesis (1). We consider CNN model of the coupled system. The dynamics of the obtained model is studied via local activity theory and edge of chaos domain is obtained for our coupled system. Computer simulations show that oscillatory patterns, static patterns, and chaotic dynamic patterns can be obtained if the parameter sets are chosen on the edge of chaos domain.

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