Graph Evolution Using Oriented Incidence Matrices

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***Abstract*—In this paper we introduce a general representation of networks that can be modelled as planar graphs, by using oriented incidence matrices. The representation is used to obtain a unitary vision on weighted and non-weighted networks corresponding to circuits that have (or do not have) passive elements between connective nodes. The implications related to the general problem of Hamiltonicity is analyzed, too.**

Keywords—Incidence matrix, Planar graphs, Weighted graphs, Hamiltonicity, Abelian group, Abelian semigroup.

I. INTRODUCTION

A variety of circuits and computer networks can be represented as planar graphs, i.e., graphs without multiple links and self-loops, and whose links do not cross each other. Modelling circuits as graph is a powerful tool: it allows one to anticipate the behavior of the modelled network, compute circuit functions, etc. In this work we provide tools that allow for a better understanding of such circuits, through oriented incidence matrices. Essentially, the results that we derive (specifically our interpretations of our new representation, and the properties derived from it) will hold to any circuit or network that can be drawn as a graph of that description.

We specifically refer to networks that can be represented as planar graphs **G(V,E)** with nodes **V**=$\{v_1,\ldots,v_N\}$ and edges **E**=$\{e_1,\ldots,e_L\}$. The graph is connected and admits a cycle representation. That means that a cycle-edge incidence matrix **B(E,L)** -- which we just call an incidence matrix -- exists.

Our main contribution in this paper is to define a common representation of the incidence matrix using directed edges and to interpret that representation in return. We furthermore define the content of the incident matrix and show how this structure reflects various interesting network properties, which will allow for network expanding or collapsing.

The paper is organized in several paragraphs as follows:

II. Basic definitions and known results
III. Introducing the cycles-edges incidence matrix
IV. The weighted oriented incidence matrix
V. Expanding graphs
VI. Conclusions

II. BASIC DEFINITIONS AND KNOWN RESULTS

In this paper we refer to planar, simple graphs: loop-free graphs that can be drawn in a plane and feature only at most single edges between nodes. The nodes and the edges are randomly indexed. The number of edges leaving and arriving in a node is called the *degree* of the node. A link connected to a node is said to be *incident* to the node. A suite of edges connecting two end nodes by passing through other nodes is called a *path*. A closed path, i.e., a path starting and ending in the same node is called a *cycle.* A cycle that visits every node of the graph exactly once is called a *Hamiltonian* cycle. A graph that has at least one Hamiltonian cycle is called a Hamiltonian graph.

A notion that is crucial to our work is that of adjacency. We recall the following definitions from [7,8]:

Definition 1: Two nodes are *adjacent* if, and only if, there is an edge connecting them.

Definition 2: Two cycles are *adjacent* if, and only if, they share precisely one common edge.

III. INTRODUCING THE ORIENTED INCIDENCE MATRIX

To exemplify our techniques, we begin by considering an undirected pentagon (in which single edges exist between the 5 nodes (which are labelled 1,…,5). We transform this graph into a directed graph, by associating random directions between the nodes. In Fig. 1 we associate each undirected pentagon edge with a tuple of two directed edges, going in opposite directions (one continuous, one dotted). In the final graph, only one of the two edges will actually exist between the nodes. The continuous directed arrows make up cycle C_i, while the dotted ones make up the cycle denoted as $-C_i$.

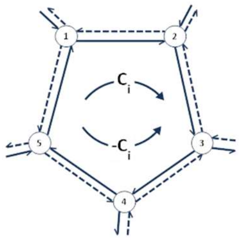


Fig. 1 A directed pentagon

Cycle representation rules (CRR). In order to parse a graph into a matrix, we will consider its nodes, its edges, and its cycles. We will associate all the internal graph cycles with one orientation (clockwise for instance), while the border cycle is parsed in the opposite direction (anti-clockwise). That is to say, we would choose either C_i or $-C_i$ for the graph in Fig. 1, but not both.

Prior work [5,6,13] has shown that each cycle can be associated with a vector C_{i1xE} of length E (the size of the edge set) whose entries C_{ik} are 1 if the cycle and the arrow k are parsed in the same direction, they are -1 if the parsing sense is different, and 0 if the cycle does not include the arrow at all. In this work, we represent 0 as ∞ with the same meaning i.e., "not connected/not adjacent".

For example, for cycle C_1 in Fig.1, if we assume that the edges are all the continuous lines, the cycle vector would be $\{\ldots,\infty,\infty,1,1,1,1,1,\infty,\infty,\ldots\}$, while $-C_1$ is associated with the vector $\{\ldots,\infty,\infty,-1,-1,-1,-1,-1,\infty,\infty,\ldots\}$.

Assume now that, for a graph G, we have associated each cycle with such vectors. We construct the CxE cycle-edge incidence matrix with rows corresponding cycles, and columns associated edges. This matrix, denoted **B(C,E)**, is a list of cycle-vectors.

Lemma 1: In simple, planar graphs with the cycle notation introduced above, each column of the matrix B(C,E) contains ∞ values except in two entries, one of which is 1, and the other, -1.

Proof: This is because of how cycle orientations are chosen, and because the graph is planar. In particular, each edge is on two cycles, one which travels it in the direction of the edge, and the other, in the opposite direction, yielding Lemma 1. □

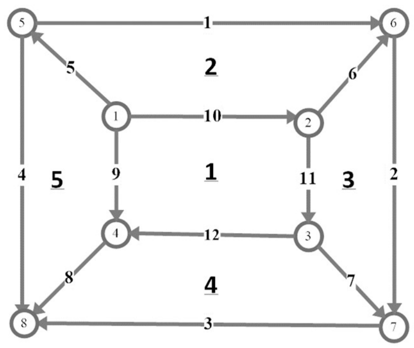


Fig.2 The directed cube graph

In Fig. 2 we depict a weighted cube graph and denote its cycles in bold font (oriented clockwise for internal cycles and anti-clockwise for the external one). Its **B(C,E)** matrix is shown in Table 1, where we underline the cycle indexes to distinguish them from the edges.

Table 1 The cycles-edges incidence matrix of a cube

	1	2	3	4	5	6	7	8	9	10	11	12
1	∞	∞	∞	∞	∞	∞	∞	∞	-1	1	1	1
2	1	∞	∞	∞	1	-1	∞	∞	∞	-1	∞	∞
3	∞	1	∞	∞	∞	1	-1	∞	∞	∞	-1	∞
4	∞	∞	1	∞	∞	∞	1	-1	∞	∞	∞	-1
5	∞	∞	∞	-1	-1	∞	∞	1	1	∞	∞	∞
6	-1	-1	-1	1	∞	∞	∞	∞	∞	∞	∞	∞

Prior work has shown that circuit operations can be mapped to graph operations on other types of incidence matrices [1-3]; in particular [4,7-11,16] shows how *cycle mergers* can be used to find Hamiltonian circuits. In [10,13] cycle merger is the operation of removing the common edge in two adjacent cycles, thus making a larger cycle.

In our notation, such merger would equate to "adding" the two rows of the incidence matrix that are associated with the initial, adjacent cycles, as defined below.

Definition 3: Assume the existence of a graph associated with a cycle-edges incidence matrix as described in our work. For cycles $\mathbf{C_i}$, $\mathbf{C_j}$, we abbreviate as $\mathbf{C_i}+\mathbf{C_j}$ the operation of "adding" their corresponding entries C_{ik}, C_{jk}, for each $k \in \{1, \ldots E\}$, as follows: ∞+∞=∞; 1+∞=1; -1+∞=-1; 1+(-1)=∞. The operation is commutative by definition.

Definition 4: A zero-cycle **ZC** is represented by a vector with E entries, all equal to ∞.

Consider now cycles C_i and $\overline{C_i} := -C_i$ from Fig.1. Clearly, adding these two cycles yields a zero-cycle.

Lemma 2: $C_k+\overline{C_k}$ yields a zero-cycle.

Lemma 3: Adding any two cycle-vectors defined as above will yield the symmetric difference [1,2] of the two vectors.

Proof. Clearly, if both cycles feature ∞ at a particular position, the addition will yield ∞+∞=∞. If exactly one entry is ∞, then the two cycles are different in that position, and this difference is preserved by the addition operation. If both cycles feature the same symbol (1 or -1), then they are similar in this position, and their similarity is preserved. If the two entries contrast (1 and -1), then the result is ∞, and the addition removes the edge from the graph. □

Theorem 1: Consider the matrix **B(N,E)** of cycle-vectors of a directed planar graph G(N,E). The set of composing cycle-vectors of **B(N,E)** have the following properties under the + operation defined before: the addition is commutative, associative, has an identity element, and an inverse element, but is not closed under addition.

Proof. We will make use of Lemma 1, which essentially states that only two cycle-vectors can have non-∞ entries: one entry is 1, the other, -1. We prove the following properties:

Commutativity: by definition.

Associativity. Let us consider cycles C_h, C_i, C_j, and their cycle-vectors, whose k-th position entries are C_{hk}, C_{ik}, C_{jk}. If C_{hk}=∞, then: $(C_{hk}+C_{ik})+C_{jk}= C_{ik}+C_{jk}$=∞+$(C_{ik}+C_{jk})$ -- so associativity holds. Now let C_{hk}=1. If $C_{ik} = C_{jk}$=∞, associativity holds as above, by commutativity. The remaining case is: either C_{ik}=-1 and C_{jk}=∞, or vice versa. For the first case: (1+ (-1))+∞=∞+∞=∞, and 1+((-1)+∞)=1+(-1)=∞. For the second case: (1+∞)+(-1)=1+(-1)=∞, while 1+(∞+(-1))=1+(-1) =∞.

Identity element. The identity element is the zero-cycle (∞, …, ∞)

Inverse element. Each entry C_{ik} has an inverse value: for 1, it is -1, for -1, it is 1, and for ∞, it is ∞. Since cycle addition relies on entry addition, the inverse of a cycle is composed of the inverses of its entries. □

Lemma 4: The result adding all the cycle-vectors of a graph is a set of disconnected nodes.

Proof. This holds specifically because the outer cycle has the inverse orientation from all the other cycles, which means that the addition of all the cycles will yield the zero-cycle (meaning the nodes are disconnected). □

IV. THE WEIGHTED ORIENTED INCIDENCE MATRIX

Directed planar graphs can model a broad category of circuits; however, accounting for passive elements, such as impedances, require the addition of *weights* to the edges [6]. Circuit analysis can then employ the *weighted matrix*.

Definition 5: The weighted matrix $\mathbf{W_{(ExE)}}$ is a square, diagonal matrix (all elements are 0 apart from the entries on the main diagonal), whose diagonal entries equal the weight of the respective edge.

The weighted incidence matrix **WB** is then obtained from the cycle-edge incidence matrix and the weighted matrix as in equation 1.

$$\mathbf{WB=BxW}, \quad (1)$$

with the proviso that 0*∞:=∞ by definition. For example, the weighted incidence matrix for the cube graph is shown in Table 2.

Table 2 Weighted incidence matrix of the cube graph

	1	2	3	4	5	6	7	8	9	10	11	12
1	∞	∞	∞	∞	∞	∞	∞	∞	$-w_9$	w_{10}	w_{11}	w_{12}
2	w_1	∞	∞	∞	w_5	$-w_6$	∞	∞	∞	$-w_{10}$	∞	∞
3	∞	w_2	∞	∞	∞	w_6	$-w_7$	∞	∞	∞	$-w_{11}$	∞
4	∞	∞	w_3	∞	∞	∞	w_7	$-w_8$	∞	∞	∞	$-w_{12}$
5	∞	∞	∞	$-w_4$	$-w_5$	∞	∞	w_8	w_9	∞	∞	∞
6	$-w_1$	$-w_2$	$-w_3$	w_4	∞	∞	∞	∞	∞	∞	∞	∞

As a sanity-check, note that if all the weights are 1, then the weighted incidence matrix is the same as the cycle-edges incidence matrix. Similarly, note that, just as the cycle-edges incidence matrix B(N,E) had only two non-∞ entries per column, one which is equal to 1, and the other, to -1, in the same way the weighted incidence matrix contains only two non-∞ entries per column, one equal to the weight on the edge indexed by that column, and the other, equal to negative the weight.

We proceed to adapt the arithmetic developed for the matrix B(N,E) in the previous section to an arithmetic for the WB matrix. Intuitively, adding two rows of the matrix will correspond to a merger of adjacent nodes, as shown in Fig. 3.

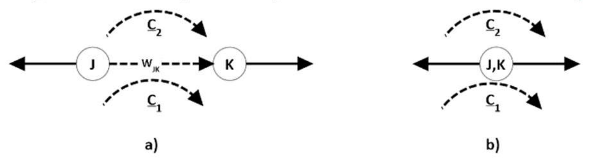


Fig. 3 Edge collapsing

In Fig. 3, nodes J and K are adjacent via a weighted, directed edge. We want addition to reflect the fact that merging these adjacent nodes will "remove" the weighted edge, making the nodes collapse. In the WB matrix, this will be reflected into a modification of some of the columns, which will no longer contain our two non-∞ entries of w and -w, but rather, two non-∞ entries equal to 0, as defined below.

Definition 6: For two entries WB_{ik} and WB_{jk}, we define $WB_{ik}+WB_{jk}$ as follows: $w_k+(-w_k):=0$, $0+\infty=0$; $\infty+\infty = \infty$; $0+0=0$; $w_k+\infty=w_k$; $- w_k +\infty=- w_k$. We furthermore require commutativity. Adding two rows of the matrix WB corresponds to an entry-by-entry addition.

Definition 7: A weighted zero-cycle **ZC** is a vector of length E, whose entries are either ∞ or 0.

Lemma 5 (cycle collapsing): Consider a directed, planar, weighted graph G(N, E), and let C represent an arbitrary cycle in this graph, reflected in some row WB_k in the weighted incidence matrix. Let (-C) represent the cycle travelling across the same edges, only in opposite direction (reflected in an entry $-WB_k$). Then:

- $WB_k + (-WB_k)$ is a weighted zero-cycle
- The operation corresponding to the addition $WB_k + (-WB_k)$ makes cycle C collapse in a single node

Proof. For the first of these notions, note that WB_k is a vector of length E, which has entries equal to ∞ except for the entries corresponding to the weighted, directed edges the cycle consists of. The vector $-WB_k$ has the same ∞ entries as WB_k, while for the non-∞ entries, $-WB_k$ features the additive inverse of those entries (since the edge is travelled in the opposite direction). As a result, the addition yields a vector of length E which has ∞ entries almost everywhere, except for the former edges of the cycle, which have collapsed in a single node as in Fig 3. □

We note that the addition operation described in Definition 6 is also associative and commutative.

Lemma 6 (Graph collapsing): Let **WB** represent the weighted incidence matrix presented here, for a directed, weighted planar graph G(N,E). The merger corresponding to the addition of all the rows of this matrix will yield a graph consisting of a single node.

Proof. This follows directly from the observation that in each column of **WB** there are only ∞ entries apart from a w_k and a $-w_k$ entry. Adding the two rows with those two non-∞ entries will yield a collapse of the edge into a single node. By repeatedly performing this operation, the result will eventually be a single node. □

V. EXPANDING GRAPHS

In this section we extend the notions of cycle-vectors, weights, and incidence to include that of circuit length. Consider a circuit C in a weighted, directed graph G. Its length λ_C is defined as:

$$\lambda_C = \sum_{k=1}^{E} |WB_{Ck}|, \qquad (2)$$

i.e., the sum of the absolute values of the weights included in a single circuit.

We have seen before that a cycle, or more generally an entire graph, can collapse into a node. The manipulation we show in this section allows for the modelling of graph expansion. Consider a weighted, directed graph with N nodes and E edges, described by its **WB** matrix.

Definition 9: A node in a graph is a cycle of length zero.

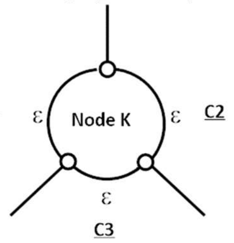
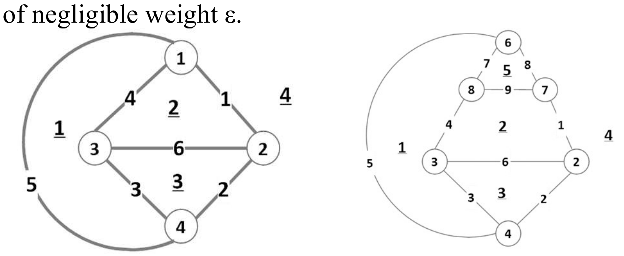


Fig. 4 A node as a cycle

Fig. 4 depicts a node, perceived as a cycle with three edges of negligible weight ε.



a) b)

Fig. 5 Graph extension

It was shown in [10,16] that graph expansion can be achieved in a controlled way by replacing nodes with such small cycles. Consider the tetrahedron in Fig. 5a, which has been expanded by replacing node 1 by a triangle, creating the new cycle 5. The values on the edges are labels, rather than weights. The transformation makes the original incidence matrix in Table 3 become the matrix in Table 4.

Table 3 The incidence matrix of the Tetrahedron

	1	2	3	4	5	6
1	∞	∞	1	1	-1	∞
2	1	∞	∞	-1	∞	1
3	∞	1	-1	∞	∞	-1
4	-1	-1	∞	∞	1	∞

Table 4. The incidence matrix of the extended Tetrahedron

	1	2	3	4	5	6	7	8	9
1	∞	∞	1	1	-1	∞	ε	∞	∞
2	1	∞	∞	-1	∞	1	∞	∞	-ε
3	∞	1	-1	∞	∞	-1	∞	∞	∞
4	-1	-1	∞	∞	1	∞	∞	-ε	∞
5	∞	∞	∞	∞	∞	∞	-ε	ε	ε

Assume that ε is chosen such that:

$\varepsilon * w \approx w$ (3)

for any real number w. We also define $\varepsilon * \infty = \infty$.

Nodes can be considered to be cycles of length 0. As a result, the incidence matrix **B** with **x** rows and **y** columns can define the incidence of nodes and edges or, more to the point, cycles and edges, wherein **x**-{1,2,…,n,…} and **y**={1,2,…, e, …}. If the graph has N nodes and E edges then **xy**=NE.

We define the row truncation matrix $\mathbf{RTM}_{N1xN}=[\mathbf{I}_{N1}\ \mathbf{0}_{(N-N1)}]$ i.e., a matrix with N1 rows and N-N1 columns, for which the leftmost N1 x N1 entries make up an identity matrix, and the remaining part of the matrix make up a zero matrix.

The matrix **RTM*B** has N1 rows, and **RTM** controls in fact the number of cycles that are considered in the graph. Let us further consider a column truncation matrix $\mathbf{CTM}_{Lx(L-L1)}=[\mathbf{W}_{L1}\ \mathbf{0}_{(L-L1)}]^T$. The submatrix $\mathbf{W}_{L1}$ is a square matrix **L1xL1** with its diagonal entries representing the weights of the **L1** edges that are considered in the graph and the rest of the entries being zero. The product **B*CTM** will control the number of columns of the incidence matrix **B**.

Hence, the trimmed incidence matrix **TB** of the graph is obtained as it is shown in equation 4.

TB=RTM*B*CTM (4)

For example, when we apply the above considerations to the graphs shown in Fig.5 a), b) we obtain the truncation matrices shown in Tab.6.

Tab.6 The truncation matrices

RTM

	1	**2**	**3**	**4**	**5**
1	1	0	0	0	0
2	0	1	0	0	0
3	0	0	1	0	0
4	0	0	0	1	0

a)

CTM

	1	**2**	**3**	**4**	**5**	**6**
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	0	1	0	0	0
4	0	0	0	1	0	0
5	0	0	0	0	1	0
6	0	0	0	0	0	1
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0

b)

In the **CTM** matrix shown in Tab. 6b, the weights of the edges are all supposed to equal 1, just for the simplicity.

Truncated matrices are useful in describing transformations of the graph, and hence, the circuit that it represents. The expansion of nodes to cycles, and the contraction of cycles to nodes is captured by matrix **TB** [4,9-11,16]**.** The latter can also be used to construct the truncated Laplacian of the graph:

$\mathbf{TL=TB*TB^T}$ (5)

If our graph describes a linear circuit, then the **TL** is the indefinite impedance matrix, which can be used for solving any linear circuit equations [3,5,6]. Trimmed matrices can also be used to solve various topological problems related to the graph, such as finding Hamiltonian circuits [6-8,11,13], or solving the travelling salesperson problem.

In fact any graph can start from an initial graph prototype and then it can be evolved using the truncation matrices; it is like a big-bang evolution. Furthermore, if the graph is cubic and planar then the evolution will preserve some basic characteristics of the initial graph, such as Hamiltonicity for example [9,16]

VI. Conclusions

This paper focuses on the modelling of circuits as planar graphs, and the subsequent quantification of circuit expansions, contractions, and modifications by means of a number of incidence matrices. We furthermore show that the evolution of graphs can be followed by trimmed and truncation matrices, which can be then be used to anticipate the properties of the new circuit, such as: the presence of a Hamiltonian circuit, the ability of the new circuit to further expand or contract, adjacency or merger of cycles in the circuit, etc.

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