An Absorbing Markov Chain Model for Stochastic Memristive Devices

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***Abstract*—In this paper we elaborate and verify a data-driven modelling approach, pertaining to the stochastic trajectory of the memristance upon the application of pulses. Our proposed approach is to model the memristor's behaviour as a time-homogeneous Markov chain. We introduce a simplified method that estimates the states and the state transition probabilities of the model from device measurements. We show that such a memristor model, generally corresponds to an absorbing Markov chain, the physical implications of which are also discussed. We apply this modelling methodology to real-world Pt/TiO2/Pt memristors and present results that validate its effectiveness in capturing the stochastic features of these devices over various timescales.**

I. Introduction

In this paper we elaborate and verify a modelling approach introduced previously, pertaining to the stochastic features of memristors. Specifically, we are interested in modelling the stochastic response exhibited by a memristor over a range of timescales, in response to a pulsed input. Figure 1 shows the measured response of a memristor to a train of voltage pulses. The long-term (stable) trend of the conductance is clearly evident. Furthermore, the inset of figure 1 shows a zoomed-in section, elaborating the stochastic and volatile nature of the trajectory over time-scales much shorter than the developing conductance trend. A model is required to capture the stochastic features of all key timescales of interest.

Previously, our proposed approach in [1] was to model the memristor's behaviour using (1), which is effectively a Hidden Markov Model:

$$\frac{ds}{dt} = \left(f_u(u,s) - \frac{s - \alpha N(t)}{\tau} + \sigma_N \eta(t) \right) f_w(s) \tag{1}$$

$$V = R(s,u)I \tag{2}$$

s is the state variable, $f_u(s,t)$ is the input sensitivity function and $f_w(s,t)$ is the window function. $\eta(t)$ is a white-noise contribution and $N(t)$ is another stochastic process discussed below. The actual resistance of the device is related to the state variable via another function $R(s,u)$.

The dynamics described by (1) for a fixed value of $\alpha N(t)$, pertain mainly to short-term characteristics of the memristor. These include a volatile change in conductance, followed by relaxation towards a stable conductance state with a characteristic time τ. These dynamics typically evolve over timescales that are of the order of microseconds to milliseconds and are referred to as the short-term or the fast timescale response. The long-term or slow timescale evolution of the (stable) conductance value on the other hand, is governed by the dynamics of the stochastic process $N(t)$ that evolves in the order of milliseconds to seconds. $N(t)$ is a random process described by a discrete state-time Markov chain. The inclusion of $N(t)$ was inspired by imposing a plausible pseudo-energy landscape, such as figure 2, on the state-space. Therefore, in (1), $N(t)$ served the purpose of shifting the minima of the parabolic landscape over time. Previously, the transition probabilities of $N(t)$ were defined as follows [1]:

$$p_{+1}(t) = Prob\left(X^+ \leq s(t) - \alpha N(t)\right) \tag{3}$$

$$p_{-1}(t) = Prob\left(X^- \geq s(t) - \alpha N(t)\right) \tag{4}$$

In this formulation the short-term dynamics and long-term dynamics are coupled. This is because the time dependant transition probabilities of the discrete Markov chain and hence the characteristics of the long-term memristor response are

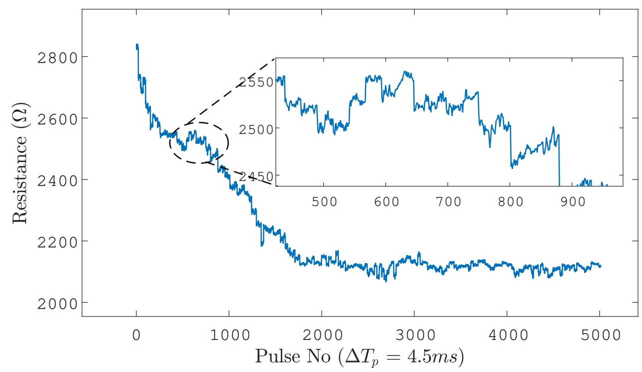

Fig. 1. Stochastic response of the memristor under constant pulsed stimulus. Every 25th pulse is a 500us wide voltage stimulus of amplitude +0.9V (200 total write pulses above). All other pulses are a read pulse. A trend of stable conductance change is visible over long timescales. The inset shows a zoomed-in section, where a stochastic meta-stable response that occurs over much shorter timescales is also evident.

computed by imagining energy barriers around the instantaneous value of $s(t)$. Although such a derivation using energy barriers is intuitively appealing and can reproduce the qualitative observations of memristor plasticity, it remains difficult to perform a quantitative fit between the long-term behaviour of the model and measurements of the actual memristors. This is because the state transition probabilities of $N(t)$ are time-inhomogeneous and implicitly linked to the complex non-linear sensitivity function $f_u(s,t)$ and the window function $f_w(s,t)$.

In this paper, we introduce a few new simplifications. Experience with actual devices has shown, that a description as elaborate as (1) may not always be necessary. In many use cases, it suffices to simulate the dynamics more coarsely, using only a discrete state-time Markov process such as $N(t)$. This provides a good fit to the measured data. Furthermore, we introduce a simplified method that estimates these discrete states and corresponding time-homogeneous state transition probabilities of the Markov chain, directly from measured device trajectories. Therefore, the remaining part of this paper will elaborate the premise of our approximations and discuss the construction of such a Markov chain. Furthermore, any physical device implications that can be inferred from the constructed model will also be explored.

II. Mathematical formulation of the model

As the premise of our fitting procedure revolves around extracting the Markov chains parameters from measured device data, we will now work directly in terms of the resistance of the device $R(N(t))$, instead of the conceptional internal state-variable $N(t)$. This is simply because we are only able to measure $R(N(t))$ directly and not $N(t)$. Thus, the modelling procedure that follows aims to approximate the continuous stochastic process $R(N(t))$ as a homogeneous discrete state-time Markov chain R_k.

We seek to appropriately discretise $R(N(t))$ and then determine the corresponding one-step transition probability matrix $\mathbf{P} = (p_{jk})$. Each element of the matrix $\mathbf{P}$ can be interpreted as the following conditional probability:

$$p_{jk} = Prob(R_{m+1} = k | R_m = j) \tag{5}$$

Therefore, $\mathbf{p}[n]$ the row vector of probability distribution of state, at time-step n, can be computed from the initial probability distribution of state $\mathbf{p}[0]$ as:

$$\mathbf{p}[n] = \mathbf{p}[0]\mathbf{P}^n \tag{6}$$

We will now discuss how to discretise the state-space and estimate the matrix $\mathbf{P}$ from measured data.

A. Computing discrete states from data

Given a measured resistance trajectory $R(t) \in (R_{min}, R_{max})$, we partition this continuous state-space Ω into a finite number M of uniformly sized disjoint sets $\{\Omega_1, \Omega_2, \Omega_3..\Omega_M\}$. If at the sampled time-step $R(t = kT_s) \in \Omega_i$, then the discrete Markov chain R_k is said to be in state i. The discrete resistance associated with set Ω_i is equal to $(R_{min} + i * \frac{R_{max} - R_{min}}{M})$. Using this procedure we attain a discrete-state trajectory for every measured trajectory $R(t)$, at a chosen time resolution. The time resolution is either equal to the sample rate of the original data or is scaled by an additional decimation factor.

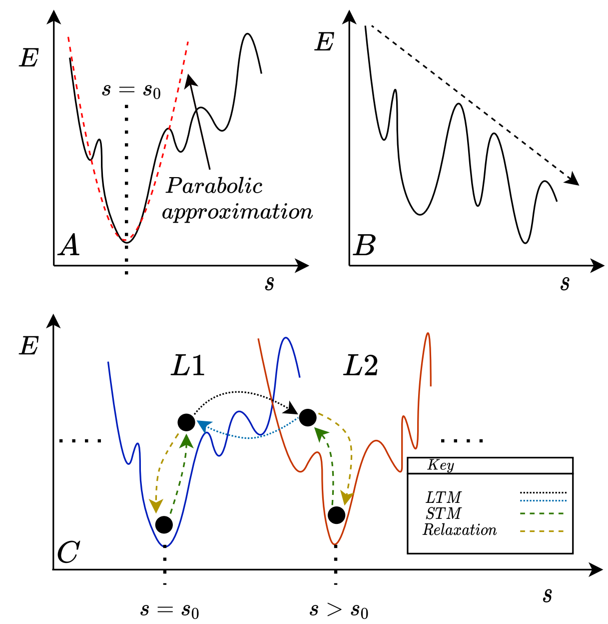

Fig. 2. **A)** an exemplar landscape under no bias. **B)** distorted landscape under bias. **C)** Volatility and plasticity can be seen as a gradual shift in the underlying landscape that confines state trajectory.

B. Estimating the transition probability matrix

At the conclusion of the aforementioned procedure, we have a sequence of states (resistances) $R_0, R_1, R_2..R_k$ from which transition probabilities can be estimated. The maximum likelihood estimator of observing a sequence, under the Markov chain model can be shown to be [2]:

$$p_{jk} = \frac{N_{jk}}{\sum_{m=1}^{M} N_{jm}} \tag{7}$$

Where N_{jk} is the number of observed transitions from state j to state k.

C. Characteristics of the transition probability matrix and the discrete state-space

In this section we will elaborate the physical interpretation and structure of our model's probability transition matrix and discrete state-space. We will also discuss how it links to the energy-landscape picture shown in figure 2.

We begin by considering the simpler case of a purely deterministic model such as [3]. In such a model, a given input stimulus and an initial condition specifies a unique deterministic trajectory in the state-space $R(t)$, towards a terminal state. Upon reaching the terminal state, no further change in state occurs despite the application of input. In practice, these terminal states are defined using suitable window functions.

Furthermore, all other points in the trajectory can be classified as transient states, as the system is bound to exit them at some point in time.

Our stochastic model and its aforementioned construction, can also be understood in a very similar fashion. After approximating the matrix $\mathbf{P}$ using (7), we can group the states into a set T_k of communicating transient states. In general there will be one or more such sets $\{T_1, T_2..T_L\}$. Similarly the remaining states will be grouped into a single set C of communicating absorbing states. States grouped within any such set can be analysed as an independent Markov sub-chain. In other words, the transition probability matrix $\mathbf{P}$ will be reducible and can be written in the form of an absorbing Markov chain [4]:

$$\mathbf{P} = (p_{jk}) = \begin{matrix} & C & T \\ C & \mathbf{S_1} & \mathbf{0} \\ T & \mathbf{R_{21}} & \mathbf{Q_2} \end{matrix} \tag{8}$$

III. Model Decomposition

The irreducible submatrix $\mathbf{S_1}$, identified in (8), represents a Markov chain on the set C of absorbing states and hence describes the limiting (i.e. $t \to \infty$) behaviour of our model. We use this to explore the nature of long-term stochastic approximations implied by our model. For example, if the submatrix $\mathbf{S_1}$ is also aperiodic, then the Markov sub-chain on the set C will be ergodic. Thus, our model approximates the limiting behaviour of the memristor, under constant pulsed stimulus, as an ergodic process with a corresponding limiting distribution. Similar analysis can be applied to the submatrix $\mathbf{Q_2}$ which describes the stochastic dynamics within the sets of transient states $T = \cup_i T_i$. These sets of states describe the stochastic transient behaviour of the memristor. $\mathbf{Q_2}$ in general, will be reducible into many smaller irreducible Markov sub-chains, one for each set T_i of communicating transient states. Therefore, the transient behaviour of the model, within a specified range of simulation time, can be analysed by examining the behaviour of the relevant Markov sub-chains and the transition characteristics between them.

In summary, the construction of the matrix $\mathbf{P}$ is equivalent to partitioning the state space, by grouping communicating states into one or more transient sets and a single absorbing set. On each one of these irreducible sets, we define time-homogeneous Markovian dynamics. Furthermore, if the dynamics of a set of communicating states are ergodic, the state-trajectory when confined in that set, can be thought of as being confined in some equivalent energy landscape [5] such as such as $L1$ and $L2$ shown in figure 2. Hence, in this approximation we maintain the existence of some time-dependant (shifting) energy landscape as proposed in [1], however the exact details of this are not elaborated. Instead, these details are estimated directly from actual device measurements and are therefore contained within the very structure of the probability transition matrix. The aforementioned decomposition is depicted in figure 3.

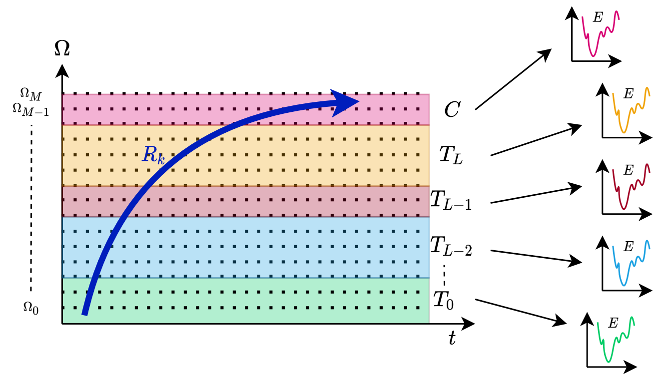

Fig. 3. The proposed stochastic process R_k, is defined by partitioning the continuous state space Ω into discrete states Ω_k and then further grouping these discrete states into sets T_i of transient states and a set C of absorbing states. Dynamics within each set are Markovian and can also be interpreted as transitions in an equivalent pseudo-energy landscape.

IV. Results

We begin this section with a discussion on how we intend to model a single device, over a range of different operating conditions. Our experiments have shown that our memristors, under a train of pulses, exhibit behaviour that is always qualitatively similar to the trajectory shown in figure 1. However, depending on the nature of the input pulses, the characteristic timescale of the transient phenomena and the limiting quasi-stationary distribution of states can change. Therefore, presently, we propose a protocol where such a model (i.e. a matrix $\mathbf{P}$) is constructed, for each different stimulus condition of interest.

We now apply this model to an actual device and show the results of the model for two different stimulus. Both examples below utilise the same device, consisting of a Pt/TiO2/Pt material stack-up as described in [6].

A. Positive Stimulus:

In this example, we construct a model for the response of the memristor to 500us wide +0.9V pulses, applied every $112.5ms$. All other pulses are read pulses. Following the aforementioned fitting procedure yields $\mathbf{P} \in \mathbb{R}^{50\times 50}$. An instance of the resulting model is simulated and superimposed on the measured trajectory in figure 4.

Decomposing $\mathbf{P}$ according to (8), we find that $\mathbf{S_1} \in \mathbb{R}^{13\times 13}$ and $\mathbf{Q_2} \in \mathbb{R}^{37\times 37}$. Therefore, the quasi-stationary distributing of the memristor, under this stimulus, is estimated using only 13 states. Similarly, 37 states are used to capture the transient behaviour of the memristor. For example, a subset of 7 communicating transient states model the plateauing phenomena seen in the data, approximately between pulse 50 to pulse 1000.

Figure 5 shows a comparison between the stationary distribution of $\mathbf{S_1}$ (and hence $\mathbf{P}$), and the distribution of the measured data from pulse 1600 on-wards (approximately ignoring the transient behaviour).

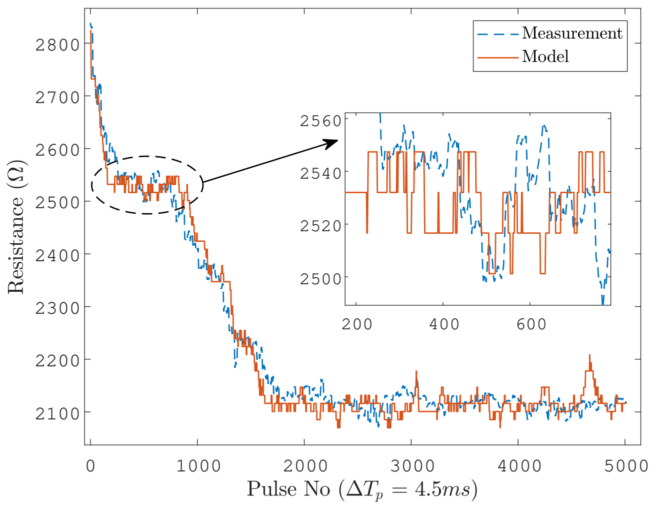

Fig. 4. Stochastic response of the memristor to 200 write pulses and a realisation of its model, starting from the same initial condition. Every 25th pulse is 500us wide +0.9V write pulse, all other pulses are +0.5V read pulses. The model consists of 50 total states grouped into 37 transient states and 13 absorbing states.

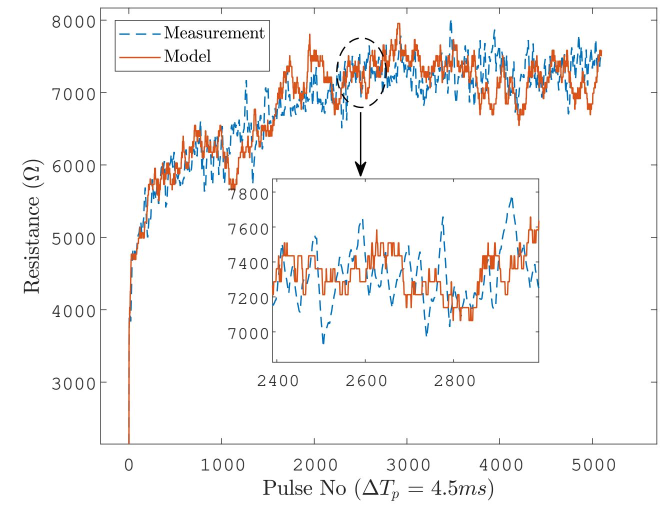

Fig. 6. Stochastic response of the memristor to 230 write pulses and a realisation of its model, starting from the same initial condition. Every 25th pulse is 500us wide -1.1V write pulse all other pulses are a real pulse. The model consists of 60 total states grouped into 19 transient states and 41 absorbing states.

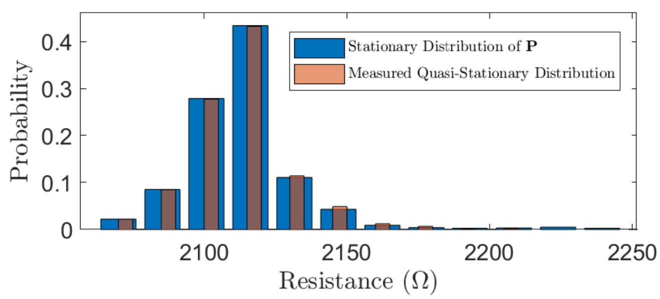

Fig. 5. A comparison between the limiting distribution of the model and the distribution of measured data of figure 4 from pulse 1600 onwards.

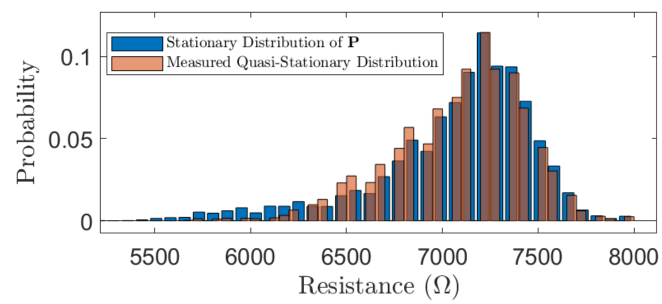

Fig. 7. A comparison between the limiting distribution of the model and the distribution of measured data of figure 6 from pulse 1100 onwards.

B. Negative Stimulus:

In this example, we construct a model for the response of the memristor to 500us wide -1.1V pulses, applied every 112.5ms. These pulses were applied immediately at the conclusion of the previous experiment. An instance of the resulting 60 state model is simulated and superimposed on the measured trajectory in figure 6. Figure 7 shows a comparison between the stationary distribution of $\mathbf{P}$, and the distribution of the measured data from pulse 1100 on-wards.

V. Conclusion

In this paper we have presented a model suitable for capturing the stochastic features of memristive devices over multiple different timescales. We have shown that the complex switching and saturating behaviour of memristive devices, can be described as probabilistic transitions between finite sets of discrete states. This approach is not only a more accurate description of these devices, but may be more computationally efficient than existing deterministic approaches, as it does not require the use of non-linear window functions. The ability to efficiently capture these stochastic features is critical for realistic simulations of memristive applications.

References

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