# Accuracy Improvement of Instantaneous Frequency Estimation by Finite Order FIR Hilbert Transformer Using Notch Filter

Keisuke Takao*, Takahiro Natori*, Toma Miyata† and Naoyuki Aikawa*
*Tokyo University of Science, Tokyo, Japan
Email: 8116049@ed.tus.ac.jp
†Salesian Polytechnic, Tokyo, Japan

***Abstract*—Frequency estimation is used in a wide range of areas such as radar, telecommunications, voice analysis and weight measurement using tuning fork sensor. This paper describes an instantaneous frequency (IF) estimation method. The original signal combined with the Hilbert transform form a complex analytic signal. The phase of the analytic signal is defined as the instantaneous phase and the time derivative of its phase IF. However, the amplitude of the signal obtained by phase-shifting the input signal by 90 degrees is different from that of the input signal because a ripple is generated in the frequency characteristics of the finite-order finite impulse response (FIR) Hilbert transformer (HT). As a result, errors are contained in the IF. In this paper, we propose a highly accurate IF estimation method using a notch filter. It is shown analytically that a harmonic frequency component is included in the IF obtained by using a finite-order FIR HT. Then, it is shown that the accurate instantaneous frequency can be obtained by removing the harmonic frequency component using an infinite impulse response (IIR) notch filter. Finally, the effectiveness of the proposed method is shown in simulations.**

***Index Terms*—frequency estimation, instantaneous frequency, Hilbert transform, notch filter**

## I. Introduction

The Hilbert transform is widely used in applications in telecommunications, mechanical engineering and medical engineering, [1] [2]. In particular, the instantaneous frequency (IF) estimation method using the Hilbert transformer (HT) is used for weight measurement using tuning fork sensor [3]. The HT changes the phase by $\pi/2$ without changing the amplitude of the original signal. The original signal combined with the Hilbert transform forms a complex analytic signal. The IF is obtained by time-differentiating the phase of the analytic signal consisting of the original signal and its Hilbert transform [4].

There are many different implementations of the FIR HT [5]–[7]. Pei and Shyu [5] presented a HT based on an eigenfilter, and Kollar et al. [6] presented a HT based on the least squares and the minimax criterion. Lim et al. [7] introduced a method for synthesis of very sharp HT using a frequency-response masking technique. These HTs can be implemented as a digital filter with finite order in actual systems. Therefore, the frequency characteristics of the obtained FIR Hilbert transformer contains ripples. As a result, the amplitude of the signal obtained by phase-shifting the input signal by 90 degrees using the FIR HT is different from that of the input signal. Thus, since the estimated IF includes harmonic frequency components, an accurate IF cannot be obtained. To obtain a more accurate IF, the order of the filter may be increased to reduce ripples in the HT. However, increasing the filter order increases the delay time. This becomes a problem in applications where fast IF estimation is required.

In this paper, we propose a highly accurate IF estimation method using a notch filter. It is shown analytically that a harmonic frequency component is included in the IF obtained by using a finite-order FIR HT. Then, it is shown that an accurate estimate of the instantaneous frequency can be obtained by removing the harmonic frequency component with a notch filter. Finally, the effectiveness of the proposed method is shown in simulations.

## II. IF estimation using finite-order Hilbert transformer

The HT is a digital filter that changes the phase of the input signal by $\pi/2$ without changing the amplitude of the input signal. The ideal frequency response of the HT is given by

$$D(e^{j\omega}) = \begin{cases} -j & (0 < \omega < \pi) \\ 0 & (\omega = 0) \\ j & (-\pi < \omega < 0). \end{cases} \tag{1}$$

When the HT is realized as an N-order linear phase FIR filter, the zero phase amplitude characteristic of its frequency response $H_0(e^{j\omega})$ is expressed as

$$H_0(e^{j\omega}) = \begin{cases} 2\sum_{n=0}^{\frac{N-2}{2}} h(n)\sin\{(n-\frac{N}{2})\omega\} & (N:\text{even}) \\ 2\sum_{n=0}^{\frac{N-1}{2}} h(n)\sin\{(n-\frac{N}{2})\omega\} & (N:\text{odd}). \end{cases} \tag{2}$$

In this paper, the filter coefficient, $h(n)$, is obtained using the Remez algorithm [8].

Now, we consider the HT with order $N = 42$ or 60, low passband edge $f_L = 0.05$, and high passband edge $f_H = 0.95$. The amplitude characteristic of the HT obtained for each order is shown in Fig. 1. It is clear from Fig. 1 that increasing the filter order reduces the size of the ripple. The HT of the input

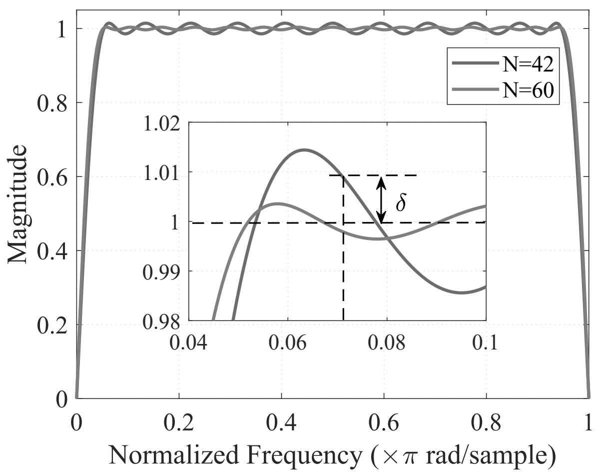


Fig. 1. The amplitude characteristic of the Hilbert transformer with finite order.

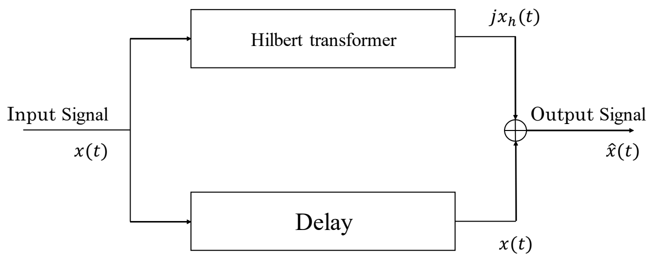


Fig. 2. The system diagram of the analytic signal generated using the Hilbert transformer.

signal $x(t)$ produces a 90 degree phase shifted signal, $x_h(t)$. Thus, the complex signal $\hat{x}(t)$ obtained by the system shown in Fig. 2 becomes

$$\hat{x}(t) = x(t) + jx_h(t). \tag{3}$$

The angle of the complex signal $\hat{x}(t)$ in (3) has the following instantaneous phase,

$$\phi(t) = \arctan \frac{x_h(t)}{x(t)}. \tag{4}$$

Moreover, the instantaneous frequency obtained by the time derivative of the phase shown in (4) is defined as

$$\tilde{f}(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt}. \tag{5}$$

As a specific example, the input signal and the 90 degree phase shifted signal obtained for each order of the HT are shown in Fig. 3 and an enlarged view is shown in Fig. 4. Here, the input signal is a sine wave with an amplitude of 1 and a frequency of 1,800 Hz. It can be seen from Fig. 4 that the signal after the HT has a different amplitude than the original signal due to the magnitude of the HT ripple at the frequency of the input signal. The IF estimated using the HT in Fig. 1 is shown in Fig. 5. It is clear from Fig. 5 that the estimated IF oscillates

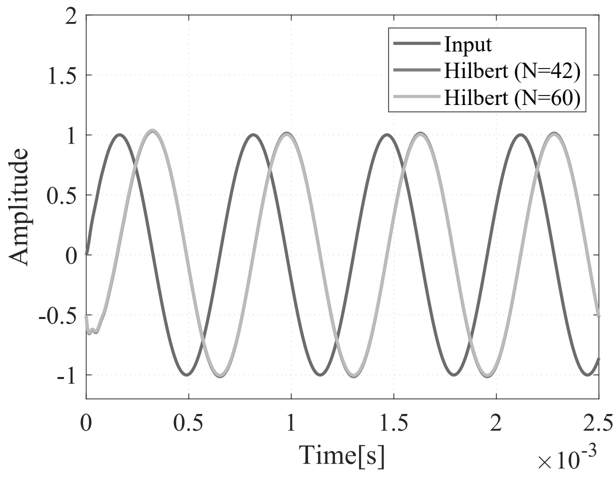


Fig. 3. Input signal and HT signal.

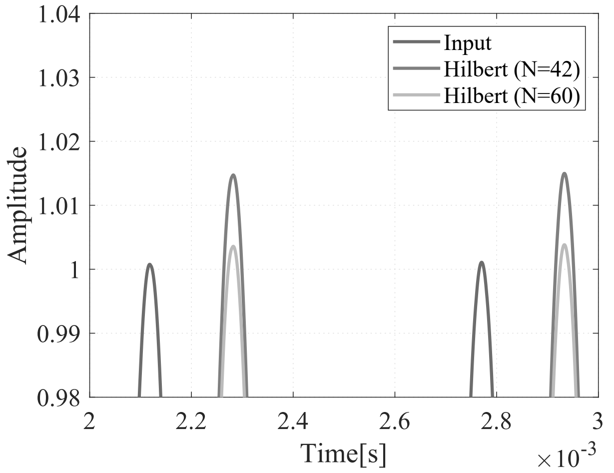


Fig. 4. Enlarged view of Fig. 3

around the true frequency. Since this vibration component is the error due to ripple, the higher the order of the HT, the higher the estimation accuracy.

Now, we consider in detail the estimated IF. We denote the input signal in Fig. 1 as

$$x(t) = A \sin \omega_1 t, \tag{6}$$

where $A$ and $\omega_1$ are the amplitude and angular frequency, respectively. If the difference in amplitude between the Hilbert transform signal and the input signal is $A\delta$ because of the finite order HT, the Hilbert transform signal, $x_h(t)$, becomes

$$x_h(t) = -A(1-\delta) \times \cos \omega_1 t. \tag{7}$$

When (6) and (7) are substituted into (4), the instantaneous phase is

$$\phi(t) = \arctan \left( \frac{-(1-\delta)\cos \omega_1 t}{\sin \omega_1 t} \right). \tag{8}$$

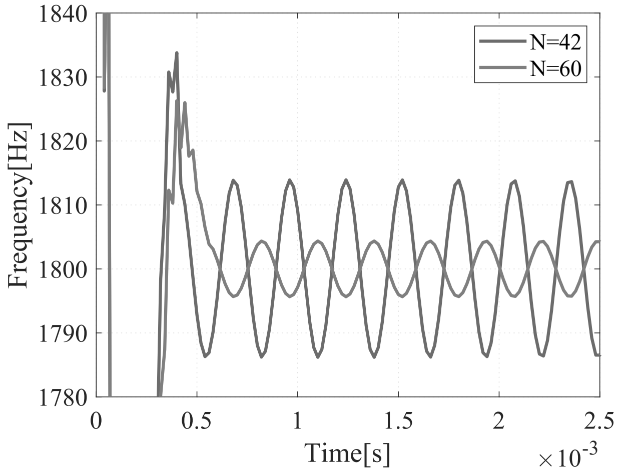


Fig. 5. The IF estimated using the finite-order Hilbert transformer.

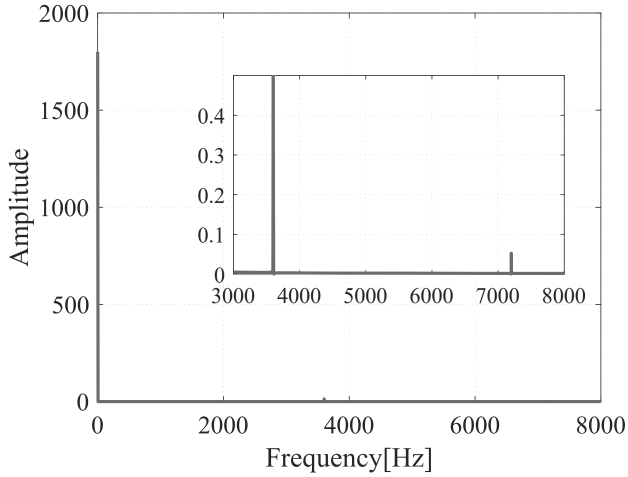


Fig. 6. The Fourier spectrum for the IF ($N = 42$).

The instantaneous angular frequency obtained by the time derivative of (8) is

$$\begin{aligned}\tilde{\omega}(t) &= \frac{d\phi(t)}{dt} \\ &= \frac{2(1-\delta)\omega_1}{1+(1-\delta)^2+[(1-\delta)^2-1]\cos(2\omega_1 t)}.\end{aligned} \quad (9)$$

When $\delta = 0$, the instantaneous angular frequency is equal to the input angular frequency $\omega_1$. However, when $\delta \neq 0$, the instantaneous angular frequency is not constant. Therefore, the estimated instantaneous frequency oscillates around the frequency of the input signal as shown in Fig. 5.

To analyze the vibration component, we consider the Fourier series expansion of $\tilde{\omega}(t)$;

$$\tilde{\omega}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\omega_1 n t), \quad (10)$$

where

$$a_n = \frac{4\omega_1}{\pi}\int_0^{\frac{\pi}{2\omega_1}} \tilde{\omega}(t)\cos(2\omega_1 n t)dt. \quad (11)$$

Here, the Fourier coefficients $a_n$ are

$$\begin{aligned} a_0 &= 2\omega_1 \\ a_1 &= \frac{2\omega_1\delta^2}{1-(1-\delta)^2} \\ a_2 &= \frac{2\omega_1\delta^4}{\{1-(1-\delta)^2\}^2} \\ &\cdots \end{aligned} \quad (12)$$

Therefore, the instantaneous angular frequency's true value consists of a DC component and vibration components that are harmonics of even multiples of the original angular frequency. Thus, the estimated IF also has vibration components that are harmonics of even multiples of the frequency of the input signal.

To confirm the above statement, the Fourier spectrum of the estimated IF in Fig. 5 is shown in Fig. 6. It can be seen that the IF has a component with frequency $3,600\,\text{Hz}$, which is twice the frequency of the input signal, and another component with frequency $7,200\,\text{Hz}$, which is four times the frequency of the input signal. Therefore, if this harmonic component can be removed with some method, an accurate IF can be obtained even with a finite order HT.

## III. Simulation

Here, we show that the estimation accuracy of the IF can be improved by removing these vibration components using the IIR notch filter.

In this simulation, the input signal is a sine wave with an amplitude of 1 and a frequency of 1,800 Hz sampled at a sampling frequency of $50\,\text{kHz}$, which assumes the output from the tuning fork sensor. Moreover, the low passband edge frequency of the HT is $1.25\,\text{kHz}$, the high passband edge frequency is $27.75\,\text{kHz}$, and the filter order is $42$. We use a notch filter to remove frequency components that are twice that of the input signal, which is considered to be the largest vibration component. The transfer function of the notch filter used here is given by

$$G(z) = \frac{1}{2}\left(1 + \frac{r - az^{-1} + z^{-2}}{1 - az^{-1} + rz^{-2}}\right), \quad (13)$$

where $a$ and $r$ are parameters that determine the notch frequency and the notch width, respectively [9] [10]. The notch frequency $f_n$ is expressed as

$$f_n = \frac{f_s}{2\pi}\arccos\left(\frac{a}{1+r}\right), \quad (14)$$

where $f_s$ is the sampling frequency. Figure 7 shows the amplitude characteristic of the IIR notch filter $G(z)$ when the sampling frequency is $f_s = 50\,\text{kHz}$, and the notch frequency is $f_n = 3,600\,\text{Hz}$ with $r = 0.1, 0.4, 0.9$. Figure 8 shows the estimated IF after removing a vibration component of

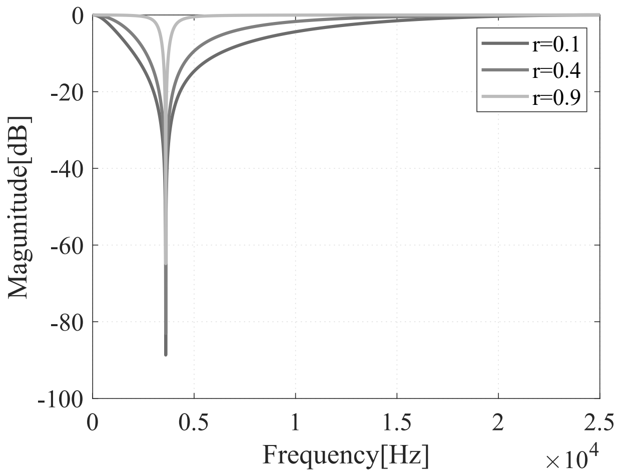


Fig. 7. The amplitude characteristic of the notch filter.

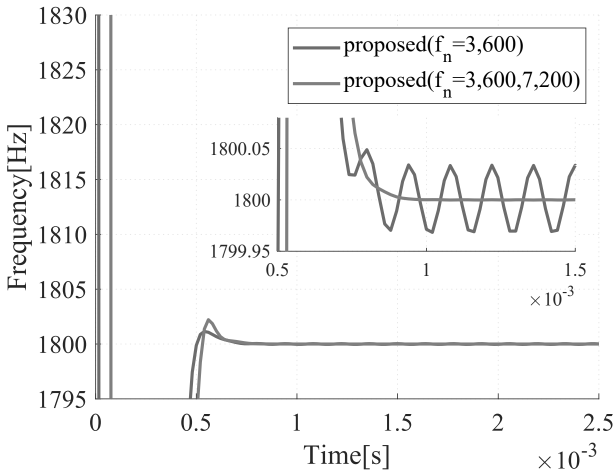


Fig. 9. The IF obtained by the proposed method ($r$=0.4).

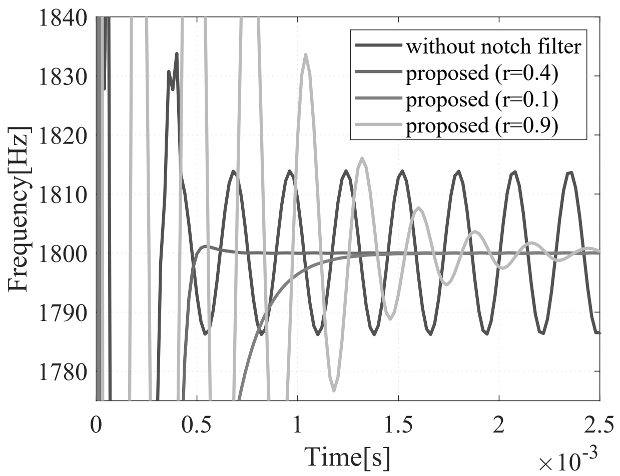


Fig. 8. The IF obtained by the proposed method using different r.

$3,600\,\mathrm{Hz}$. Although there are differences depending on the value of $r$, the estimation accuracy is improved with the proposed method. Next, the result of removing the $3,600\,\mathrm{Hz}$ and $7,200\,\mathrm{Hz}$ components is shown in Fig. 9. From this result, it can be confirmed that the accuracy of the estimation is improved by removing components whose frequency is four times that of the input signal rather than only removing components whose frequency is twice that of the input signal. To obtain the same accuracy as the proposed method using the HT alone, if frequencies twice as large as the input signal are removed, the order of the HT must be 118, and if frequencies four times as large as the input signal are removed, the order must be 190. Therefore, the proposed method can obtain high-accuracy IF with low-order HT.

## IV. Conclusion

In this paper, we theoretically showed that the IF obtained using the HT with a finite order contains harmonics. By removing the harmonics using the IIR notch filter, it was shown through an example that a high precision IF could be obtained even with low order HT.

In the future, we plan to propose a method that uses FIR filters instead of IIR notch filters.

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